

Gauss-Bonnet dark energy Chaplygin Gas Model

A. Khodam-Mohammadi*, E. Karimkhani[†] and A. Alaei

*Department of Physics, Faculty of Science,
Bu-Ali Sina University, Hamedan 65178, Iran*

Abstract

The correspondence of the Gauss-Bonnet (GB) and its modification (MGB) models of dark energy with the standard and generalized Chaplygin gas-scalar field models (SCG and GCG) have been studied in a flat universe. The exact solution of potentials and scalar fields, which describe the accelerated expansion of the universe, are reconstructed. According to the evolutionary behavior of the GB and MGB models, the same form of dynamics of scalar field and potential for different SCG and GCG models are derived. By calculating the squared sound speed of the MGB, GB model as well as the SCG, GCG, and investigating the GB-Chaplygin gas from the viewpoint of linear perturbation theory, we find that the best results which is consistent with the observation, may be appeared by considering the MGB-GCG. Also we find out some bounds for parameters.

* Email: Khodam@basu.ac.ir

[†] Email: E.karimkhani91@basu.ac.ir

I. INTRODUCTION

Astrophysical data which is out coming from distant Ia supernova [1–3], Large Scale Structure (LSS) [4, 5] and Cosmic Microwave Background (CMB)[6, 7], indicate that our universe undergoes with an accelerating expansion. This kind of expansion may be arisen by a mysterious energy component with negative pressure, so called, dark energy (DE).

However in the last decades, other models based on modified gravity ($F(R), F(G), F(R, \phi, X), F(T), \dots$) have been proposed that have given another description of acceleration expansion of the universe. In these models, many authors have showed that all models of DE can be resolved by modifying the curvature term R (Ricci scalar) of Einstein-Hilbert action with another curvature scalars such as any scalar function of R , Gauss-Bonnet term (G), torsion (T), scalar-tensor (X, ϕ) and etc. (details are in Ref. [8–21] and references there in). Even, some authors found that, the early inflation, the intermediate decelerating expansion and late time acceleration expansion, could be described together in one model [22].

Lately, among many models of DE, dynamical models, which are considering a time dependent component of energy density and equation of state, have attracted a great deal of attention. Also, among many dynamical models, ones that represented by a power series of Hubble parameter and its derivative (*i.e.* $\dot{H}, H\dot{H}, H^2, \dots$) have been interested [23, 24]. Also authors in [25–29] have shown that terms of the form $H^3, \dot{H}H^2$ and H^4 can be important for studying of the early universe. Hence, it would not be some thing strange to consider a DE density proportional to the Gauss-Bonnet (GB) term which is invariant in 4-dimensional. Besides, in geometrical meaning, the GB invariant has a valid dimension of energy density [30]. Also authors in [31, 32] showed that a unification between early time inflation and late time acceleration in a viable cosmology can be described by a coupling between GB term and a time varying scalar field [33].

The other successful model of DE is Chaplygin Gas model. The standard Chaplygin Gas model (SCG), first proposed by [34–36], regards as a perfect fluid which plays a dual role in the history of the universe: it behaves as dark matter in the first epoch of evolution of the universe and as a dark energy at the late time. Unfortunately this model has some inconsistency with observational data like SNIa, BAO, CMB [37–39]. So Generalized Chaplygin Gas (GCG) [40] and Modified Chaplygin Gas (MCG) models [41–43] have been

introduced in order to establish a viable cosmological model. It would be beneficial to study any relationship between SCG model and its modification while DE density behaves like GB invariant term as mentioned above. In this paper we would show that it leads interesting cosmological implications.

As we would show in this paper, the EoS parameter of GB DE model on its own does not give rise to phantom phase of the universe. Besides in [30], author shows that presence of matter drastically converts Friedmann equation into a nonlinear differential equation which alters the behavior of the EoS parameter which can lead to $w_o \sim -1.17$ and allows for quintom behavior. However, in this paper, we incorporate GB dark energy density with a SCG component without adding any matter content. In addition, corporation GB or MGB with different CG models (*i.e.* SCG and GCG) would be help full in order to obtain exact solution for scalar field and potential and would relieve us in order to determine some bounds for free parameters of models. So considering the cosmological solution for different compositions of GB and CG models could show the importance of each one. Also, we would succeed in the frame work where $\kappa^2 = 8\pi G = M_p^{-2} = 1$ and in the natural unit where ($\hbar = c = 1$).

The outline of this paper is as follows: In next section, we introduce the GB dark energy and calculate the deceleration and EoS parameters. Then, in subsections 2.1 , 2.2 and 2.3 we investigate corporation GB with SCG and GCG, in turn and then scalar field and scalar potential are obtained by exact solution. In section III, the same procedure has done for MGB energy density. In section IV, we would investigate Adiabatic Sound Speed, v^2 , which is one of the critical physical quantity in the theory of linear perturbation. In section V, we discuss on behavior of scalar field, scalar potential and deceleration parameter *versus* x for GB and MGB models and we gain some bounds for free parameters of models. Finally, we summarize our results in Sec. VI.

II. GAUSS-BONNET DARK ENERGY IN A FLAT UNIVERSE

The energy density of GB-DE is given by

$$\rho_D = \alpha \mathcal{G} \tag{1}$$

where α is a positive dimensionless parameter [30]. Gauss-Bonnet invariant \mathcal{G} is topological invariant in four dimensions and may lead to some interesting cosmological effects in higher dimensional brane-world (for a review, see [44]). It is defined as

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\eta\gamma}R^{\mu\nu\eta\gamma} \quad (2)$$

where R , $R_{\mu\nu}$ and $R_{\mu\nu\eta\gamma}$ are scalar curvature, Ricci curvature tensor and Riemann curvature tensor, respectively. In a spatially flat FRW universe

$$d^2s = -dt^2 + a^2(t) [dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2] \quad (3)$$

the Eq. (1) takes the form

$$\rho_D = 24\alpha H^2 (H^2 + \dot{H}). \quad (4)$$

By using the energy density ρ_D , without any matter component, the Friedmann equation in flat universe in reduced Planck mass unit ($8\pi G = \hbar = c = 1$) is

$$H^2 = \frac{1}{3}\rho_D = 8\alpha H^2 (H^2 + \dot{H}). \quad (5)$$

Defining the e-folding x with definition $x = \ln a = -\ln(1+z)$, where z is the redshift parameter and using $d/d(x) = \frac{1}{H}d/d(t)$, we get the following differential equation

$$H^2 + \frac{1}{2} \frac{dH^2}{dx} - \frac{1}{8\alpha} = 0, \quad (6)$$

which immediately gives the solution

$$H(x) = \sqrt{\frac{1}{8\alpha}(1 + \xi e^{-2x})}. \quad (7)$$

The parameter ξ is a constant of integration which is obtained by $\xi = 8\alpha H_0^2 - 1$. Also it gives $\alpha = (1 + \xi)/(8H_0^2)$. Using the continuity equation

$$\dot{\rho}_D + 3H(1 + w_D)\rho_D = 0 \quad (8)$$

and Eqs. (4),(5), the equation of state (EoS) parameter yields

$$w_D = -1 - \frac{\dot{\rho}_D}{3H\rho_D} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} = -1 - \frac{2}{3} \left(\frac{1}{8\alpha H^2} - 1 \right). \quad (9)$$

It is more preferable to write above equation in term of e-folding, x . Hence, by using Eq. (7), the EoS parameter can be rewritten as

$$w_D = -1 + \frac{2}{3} \left(\frac{\xi e^{-2x}}{1 + \xi e^{-2x}} \right). \quad (10)$$

We see that the constant ξ plays a crucial role in the behavior of the EoS parameter. For $\xi = 0$ (*i.e.* $8\alpha H_0^2 = 1$), the EoS parameter for Λ CDM model ($w_\Lambda = -1$) is retrieved. For $\xi > 0$ the expanding universe accelerates in quintessence phase ($-1 < w_D < -1/3$). Using Eqs. (5) and (7), the deceleration parameter is calculated as

$$q = -1 - \frac{\dot{H}}{H^2} = -\frac{1}{8\alpha H^2} = -\frac{1}{1 + \xi e^{-2x}}. \quad (11)$$

Since α and H_0^2 are positive parameters, so ξ always must be greater than -1 . Therefore, the deceleration parameter is always negative except for $-1 < \xi < 0$. In this way, the universe which is characterize by GB dark energy model could not exhibit a transition from deceleration to acceleration phase for $\xi \geq 0$, against what we expect from observations.

A. Gauss Bonnet Standard Chaplygin Gas

The SCG is a perfect fluid with an equation of state as

$$p_{SCG} = -\frac{A}{\rho}, \quad (12)$$

where p , ρ and A are pressure, energy density and a positive constant respectively. By substituting Eq.(12) into the continuity equation (8), the energy density immediately solved

$$\rho_{SCG} = \sqrt{A + B e^{-6x}}, \quad (13)$$

where B is an integration constant [45]. Using the standard scalar field DE model in which the energy density and pressure are defined as

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \sqrt{A + B e^{-6x}}, \quad (14)$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = \frac{-A}{\sqrt{A + B e^{-6x}}}, \quad (15)$$

and equating $p_{SCG} = p_\phi$ and $\rho_{SCG} = \rho_\phi$, the scalar potential and kinetic energy term of SCG model are given as

$$V(\phi) = \frac{2A + B e^{-6x}}{2\sqrt{A + B e^{-6x}}} \quad (16)$$

$$\dot{\phi}^2 = \frac{B e^{-6x}}{\sqrt{A + B e^{-6x}}}. \quad (17)$$

Also the EOS parameter becomes

$$w_{SCG} = \frac{p}{\rho} = -\frac{A}{A + B e^{-6x}} \quad (18)$$

Equating the energy densities (*i.e.*, $\rho_{SCG} = \rho_D$) and EoS parameters (*i.e.*, $w_{SCG} = w_D$), after using the Friedmann equation (5), constants A and B immediately given by

$$A = \frac{3}{(8\alpha)^2} [(2 + \xi e^{-2x})^2 - 1], \quad (19)$$

$$B = e^{6x} \left[\left(\frac{3}{8\alpha} (1 + \xi e^{-2x}) \right)^2 - A \right], \quad (20)$$

and hence the scalar potential and kinetic energy term rewritten as

$$V(x) = \frac{1}{8\alpha} (3 + 2\xi e^{-2x}) = \frac{H_0^2}{1 + \xi} (3 + 2\xi e^{-2x}), \quad (21)$$

$$\dot{\phi} = \frac{1}{2} \sqrt{\frac{\xi e^{-2x}}{\alpha}}. \quad (22)$$

By inserting $\phi' = \dot{\phi}/H$, where prime means derivative with respect to $x = \ln a$, the differential equation (22) gives the normalized scalar field ($\phi = 1$ at present, $x = 0$) in terms of x as

$$\phi = 1 - \frac{\sqrt{2}}{2} \ln \left(\frac{1 + 2\xi e^{-2x} + 2\sqrt{\xi e^{-2x}(1 + \xi e^{-2x})}}{1 + 2\xi + 2\sqrt{\xi(1 + \xi)}} \right). \quad (23)$$

It is easy to see that from Eq. (7), at present, we must have $1 + \xi \geq 0$ and from (23), it must be required that $\xi(1 + \xi) \geq 0$. Therefore in this model, we must have $\xi \geq 0$. As it is shown in Fig. 1, the normalized scalar field grows up to a saturated value at late time in such a way that this value exceeds for larger values of ξ . Also Eq. (21) shows that the universe goes to a stable equilibrium at infinity where $V(\infty) = 3H_0^2/(1 + \xi)$ and from (23), the scalar field reaches to $\phi(\infty) = 1 + (\sqrt{2}/2) \ln \left(1 + 2\xi + 2\sqrt{\xi(1 + \xi)} \right)$.

B. Gauss Bonnet Generalized Chaplygin Gas

The equation of state of generalized Chaplygin gas (GCG) defined as [46]

$$p = -\frac{A}{\rho^{\delta-1}}, \quad (24)$$

where A is a constant and $1 \leq \delta \leq 2$. For $\delta = 2$, it reaches to SCG model. The energy density, similar to previous case, is given by

$$\rho_{GCG} = (A + B e^{(-3\delta x)})^{\frac{1}{\delta}}, \quad (25)$$

and the scalar field model gives energy density and pressure of GCG as

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) = (A + B e^{-3\delta x})^{\frac{1}{\delta}}, \quad (26)$$

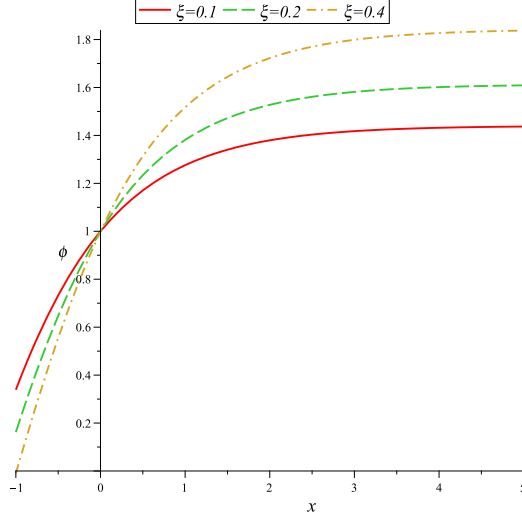


FIG. 1: behavior of normalized scalar field *versus* e-folding x for some values of $\xi \geq 0$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = -A(A + Be^{-3\delta x})^{-\frac{\delta-1}{\delta}}. \quad (27)$$

After forward calculation, three quantities: the scalar potential, kinetic term and EoS parameter are given by

$$V(x) = \frac{2A + Be^{-3\delta x}}{2(A + Be^{-3\delta x})^{\frac{\delta-1}{\delta}}}, \quad (28)$$

$$\dot{\phi}^2 = \frac{Be^{-3\delta x}}{(A + Be^{-3\delta x})^{\frac{\delta-1}{\delta}}}, \quad (29)$$

$$w_{GCG} = \frac{p}{\rho} = -\frac{A}{A + Be^{-3\delta x}}. \quad (30)$$

Also same as previous, the constants A and B reconstructed as

$$A = \frac{3 + \xi e^{-2x}}{(8\alpha)^\delta} [3(1 + \xi e^{-2x})]^\delta, \quad (31)$$

$$B = e^{3\delta x} \left[\left(\frac{3}{8\alpha} (1 + \xi e^{-2x}) \right)^\delta - A \right] \quad (32)$$

and the potential and dynamics of GB-GCG can be written as

$$V(x) = \frac{3 + 2\xi e^{-2x}}{8\alpha}, \quad (33)$$

$$\dot{\phi} = \frac{1}{2} \sqrt{\frac{\xi e^{-2x}}{\alpha}}. \quad (34)$$

As it is seen, the potential and dynamics of GB-GCG are not a function of parameter δ and are exactly similar to previous case (see Eqs. (21) and (22)). Therefore the reconstructed

scalar potential and scalar field obtained by previous Eqs. (21) and (23). It is worthwhile to mention that both models that we have been studied, encourage with an essential problem. Despite of observational predictions, the phase transition between deceleration to acceleration expansion did not happen in GB-DE model. Therefore we will study on the MGB model, which may alleviate this problem.

III. MODIFIED GAUSS BONNET DARK ENERGY

The energy density MGB has been defined by

$$\rho_D = 3H^2(\gamma H^2 + \lambda \dot{H}), \quad (35)$$

where γ and λ are dimensionless constants [30]. The Friedmann equation in dark dominated flat universe gives

$$\gamma H^2 + \frac{1}{2}\lambda \left(\frac{dH^2}{dx} \right) - 1 = 0, \quad (36)$$

and the Hubble parameter given by

$$H(x) = \sqrt{\frac{1}{\gamma}(1 + \eta e^{-\frac{2\gamma x}{\lambda}})}, \quad (37)$$

where η is an integration constant which is obtained by $\eta = \gamma H_0^2 - 1$. The EoS parameter becomes

$$w_D = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} = -1 + \frac{2\gamma}{3\lambda} \left(\frac{\eta e^{-\frac{2\gamma x}{\lambda}}}{1 + \eta e^{-\frac{2\gamma x}{\lambda}}} \right) \quad (38)$$

and deceleration parameter is obtained as follows

$$q = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{\gamma}{\lambda} \left(\frac{\eta e^{-\frac{2\gamma x}{\lambda}}}{1 + \eta e^{-\frac{2\gamma x}{\lambda}}} \right). \quad (39)$$

For positive values of γ and λ , from Eq. (37), it is easy to see that η must be always greater than -1 and from Eq. (39), a transition from deceleration to acceleration is expected provided that $\eta \geq 0$. Detailed discussion were transferred to section V.

A. Modified Gauss Bonnet And SCG

Now we want to investigate on the correspondence between MGB-SCG models and reconstruct the potential and dynamics of scalar field. Same as before, equating energy densities

(*i.e.*, Eqs. (13) and (35)) and EoS parameters (*i.e.*, (18) and (38)), yield

$$A = \frac{9}{\gamma^2} \left(1 + \eta e^{-\frac{2\gamma x}{\lambda}}\right) \left[1 + \left(1 - \frac{2\gamma}{3\lambda}\right) \eta e^{-\frac{2\gamma x}{\lambda}}\right], \quad (40)$$

$$B = e^{6x} \left[\left(\frac{3}{\gamma} (1 + \eta e^{-\frac{2\gamma x}{\lambda}}) \right)^2 - A \right]. \quad (41)$$

By substituting A and B in Eqs. (16) and (17), we find

$$V(x) = \frac{3}{\gamma} \left[1 + \left(1 - \frac{\gamma}{3\lambda}\right) \eta e^{-\frac{2\gamma x}{\lambda}}\right], \quad (42)$$

$$\dot{\phi} = \sqrt{\frac{2\eta}{\lambda} e^{-\frac{2\gamma x}{\lambda}}}, \quad (43)$$

which immediately gives the normalized scalar field as

$$\phi = 1 - \frac{\sqrt{2}}{2} \sqrt{\frac{\gamma}{\lambda}} \ln \left(\frac{1 + 2\eta e^{-\frac{2\gamma x}{\lambda}} + 2\sqrt{\eta e^{-\frac{2\gamma x}{\lambda}} (1 + \eta e^{-\frac{2\gamma x}{\lambda}})}}{1 + 2\eta + 2\sqrt{\eta(1 + \eta)}} \right). \quad (44)$$

The behavior of scalar field in this model is the similar to GB-DE model as discussed in Sec. II A.

B. Modified Gauss Bonnet And GCG

As previous, the constants A and B are

$$A = \left(\frac{3}{\gamma}\right)^\delta \left(1 + \eta e^{-\frac{2\gamma x}{\lambda}}\right)^{\delta-1} \left[1 + \left(1 - \frac{2\gamma}{3\lambda}\right) \eta e^{-\frac{2\gamma x}{\lambda}}\right], \quad (45)$$

$$B = e^{3\delta x} \left[\left(\frac{3}{\gamma} (1 + \eta e^{-\frac{2\gamma x}{\lambda}}) \right)^\delta - A \right]. \quad (46)$$

and the potential and dynamics of MGB-GCG are given by

$$V(\phi) = \frac{3}{\gamma} \left[1 + \left(1 - \frac{\gamma}{3\lambda}\right) \eta e^{-\frac{2\gamma x}{\lambda}}\right] \quad (47)$$

$$\dot{\phi} = \sqrt{\frac{2\eta}{\lambda} e^{-\frac{2\gamma x}{\lambda}}} \quad (48)$$

which are exactly similar to (42) and (43) in previous model. Therefore the behavior of normalized scalar field and potential are the same as MGB-SCG model.

IV. ADIABATIC SOUND SPEED

Investigation of the squared of sound speed, v^2 , would help us to determine the growth of perturbation in linear theory [47]. The sign of v_s^2 plays a crucial role in determining the stability of the background evolution. Positive sign of v^2 shows the periodic propagating mode for a density perturbation and probably represents an stable universe against perturbations. The negative sign of it shows an exponentially growing/decaying mode in density perturbation, and can show sounds of instability for a given model. The squared of sound speed is defined as [47]

$$v^2 = \frac{dP}{d\rho} = \frac{\dot{P}}{\dot{\rho}} \quad (49)$$

In a dark dominated flat universe, it can be written as

$$v^2 = -1 - \frac{1}{3} \left(\frac{\ddot{H}}{\dot{H}H} \right). \quad (50)$$

and it immediately gives a constant squared of sound speed for GB-DE as $v^2 = -1/3$. Therefore it may reveal an instability against the density perturbation in GB-DE model. For MGB-DE, Eq.(50) gives

$$v^2 = -1 + \frac{2\gamma}{3\lambda}. \quad (51)$$

It shows that v^2 can be positive provided that $\gamma/\lambda > 3/2$. Thus an stable DE dominated universe may be achieved in this model. In the next section we would improve this bound for γ/λ in a proper way.

V. DISCUSSION

We are interesting to focus on MGB-DE model. At first, we start with Eq. (39) and plot the deceleration parameter with respect to x in Fig. 2. It shows that the deceleration parameter transits from deceleration ($q > 0$) to acceleration ($q < 0$) in some point at the past. The parameters η and γ/λ play a crucial rule for this point. As η or γ/λ adopt bigger values, the transition point approaches to present time. By choosing the best values of q_0 (~ -0.6) and inflection point as ($x \simeq -0.5$) which has been parameterized recently [48–50], we obtain some bounds for η and γ/λ as follow

$$0 < \eta < 2.5 \quad 1.5 \leq \frac{\gamma}{\lambda} \leq 3 \quad (52)$$

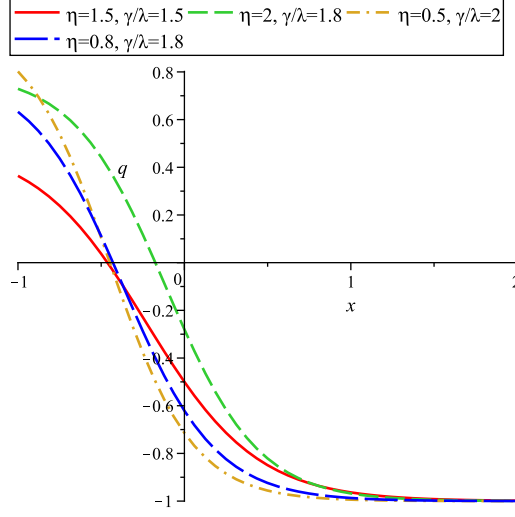


FIG. 2: The behavior of deceleration parameter q *versus* e-folding x for various η and γ/λ . The transition from deceleration to acceleration was happened around $x \sim 0.5$

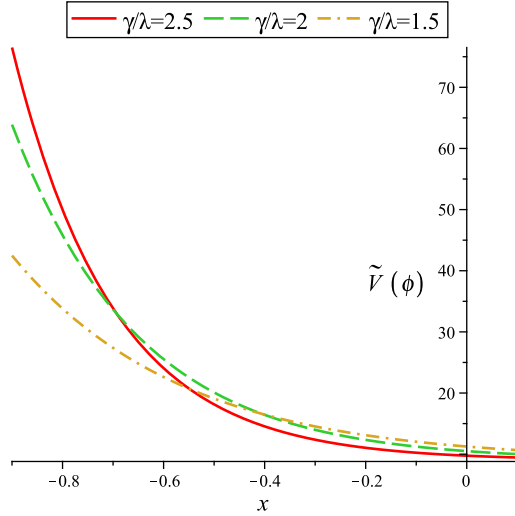


FIG. 3: behavior of $V(\tilde{\phi})$ *versus* e-folding x for various γ/λ and $\eta = 1.5$

Using Eq. (42) for MGB model, we plot $V(\tilde{\phi}) = \gamma V(\phi)$ *versus* x for different values of γ/λ and $\eta = 1.5$ in Fig. 3. This figure shows that as time goes, $V(\tilde{\phi})$ is decreasing to small values and the potential will reach to a constant at infinity. In addition, by increasing the ratio of γ/λ , the tracking potential adopts bigger values at future.

As it is seen, the potential describes a tracker solution. According to the quintessential

tracker solution, our universe undergoes a phase from $w = 0$ to $w = -1$ and the effective EoS is $w_{eff} = -0.75$ [51]. The huge advantage of the tracker solution is that it allows the quintessence model to be insensitive to initial conditions [52]. So we use this feature in order to improve obtained bounds of parameters. In this way, Eqs. (38) and (47), for matter dominated universe ($w = 0$), leads to $V(\phi) = 3/(2\gamma - 3\lambda)$. On the other hand the quantity $V(\phi)$ for quintessence barrier ($w = -1/3$) reach to $V(\phi) = 2/(\gamma - \lambda)$, so that the value $\gamma/\lambda = 1$ is illegal. It is also consistent with what we got from investigation of the deceleration parameter. Finally, the potential might give a tracking solution provided that $1.5 < \gamma/\lambda \leq 3$.

VI. CONCLUSION

In this paper, the reconstruction of GB-DE and some variety of Chaplygin gas have been studied. We obtained exact solutions for reconstructed scalar field and its potential in each models (GB-SCH, GB-MCG, MGB-SCG and MGB-MCG). According to cosmological predictions and historical evolutions, some models should be rejected (*i.e.*, models combined with GB-DE) and another models which have been combined with MGB-DE can be permitted to express the evolution of the universe. The equation of state and deceleration parameters for both GB and MGB models were calculated. In GB-DE model, the deceleration parameter was always negative except for $-1 < \xi < 0$. This fact was shown that a transition from deceleration to acceleration expansion could not have happened in the past that is contrary to the facts of cosmology. Also it was easily shown that the EoS parameter in GB-DE model would not ever reach to phantom phase (*i.e.* $w_D < -1$). We showed that for $\xi = 0$ (*i.e.* $8\alpha H_0^2 = 1$), the EoS parameter for Λ CDM model was retrieved. Investigation on the squared of sound speed, revealed an instability of model against density perturbation in GB-DE model.

In MGB-DE model, we found that the transition from deceleration to acceleration is permitted just for a limited range of values of η and γ/λ . Choosing the best values for deceleration parameter at present and deflection point, according to observations, some bounds of $0 < \eta < 2.5$ and $1.5 \leq \gamma/\lambda \leq 3$ were obtained. We showed that by redefining $\gamma V(\phi) = \tilde{V}(\phi)$, the scalar potential decreased to smaller values and will reach to a saturated constant at late time. Our investigation on $V(\phi)$ for two phases, matter dominate and

quintessence, showed that γ/λ , could not take two values 1 and 3/2.

It will be interesting to find the constraints of these models against the data of cosmological observations and structure formation. We hope to discuss these issues in the future.

-
- [1] A. G. Riess *et al.*, *Astron. J.* **116**, 1009 (1998).
 - [2] S. Perlmutter *et al.*, *Nature* **391**, 51 (1998).
 - [3] M. Hicken *et al.*, *Astrophys. J.* **700**, 1097,(2009).
 - [4] M. Tegmark *et al.*, *Astrophys. J.* **606**, 702 (2004).
 - [5] K. Abazajian *et al.*, [SDSS Collaboration] *Astron. J.* **129**, 1755 (2005).
 - [6] D.N. Spergel *et al.*, *Astrophys. J. Suppl.* **148**, 175 (2003).
 - [7] E. Komatsu *et al.*, [WMAP Collaboration], *Astrophys. J. Suppl.* **180**, 330 (2009).
 - [8] S. Nojiri and S.D. Odintsov, *Int. j. Geom. Methods M.* **4**, 115 (2007).
 - [9] I. Martino, M. Laurentis and S. Capozziello, *Universe*, **1**, 199, 2015.
 - [10] S. Capozziello *et al.*, *Universe*, **1**, 199 (2015).
 - [11] S. Bahamonde, C.G. Böhm, F.S.N. Lobo and D. Sáez-Gómez, *Universe*, **1**, 186 (2015).
 - [12] S. Basilakos, N.E. Mavromatos and J. Solà, *Universe*, **2**, 14 (2016).
 - [13] L. Iorio *et al.*, *Physics of the Dark Universe*, **13**, 111 (2016).
 - [14] Lorenzo Iorio, Ninfa Radicella and Matteo Luca Ruggiero, *JCAP*, **08**, 21 (2015).
 - [15] K. Rezazadeh, A. Abdolmaleki and K. Karami, *JHEP*, **01**, 131 (2016).
 - [16] K. Karami, A. Abdolmaleki, S. Asadzadeh and Z. Safari, *Eur. Phys. J. C* **73**, 2565 (2013).
 - [17] A. Khodam-Mohammadi, P. Majari and M. Malekjani, *Astrophys. Space Sci.* **331**, 673 (2011).
 - [18] A. De Felice and S. Tsujikawa, *living Rev. relativ.* **13**, 3 (2010).
 - [19] T.P. Sotiriou and V. Faraoni, *Rev. Modern Phys.* **82**, 451 (2010).
 - [20] A. Zanzi, *Universe* **1**, 446 (2015).
 - [21] Yi-Fu Cai, S. Capozziello, M. De Laurentis and E.N. Saridakis, *Rept. Prog. Phys.* **79**, 106901 (2016).
 - [22] P.H. Chavanis, *Universe*, **1**, 357 (2015).
 - [23] A. Gomez-Valent *et al.*, *JCAP* **01** 004(2015).
 - [24] A. Gomez-Valent *et al.*, *Mon. Not. Roy. Astron. Soc.* **448**, 2810 (2015).
 - [25] J.A.S. Lima, S. Basilakos, and J. Solà, *Mon. Not. Roy. Astron. Soc.* **431**, 923 (2013).

- [26] E.L.D. Perico, J.A.S. Lima, S. Basilakos, and J. Sola, Phys. Rev. D **88**, 063531 (2013).
- [27] S. Basilakos, J. A. S. Lima, and J. Sola, Int. J. Mod. Phys. D **22**, 1342008 (2013).
- [28] L.E. Bleem *et al.*, Astrophys. J. **216**, 27 (2015).
- [29] J.A.S. Lima, M. Trodden, Phys. Rev. D **53**, 4280(1996).
- [30] L.N. Granda, Mod. Phys. Lett. A **28**, 1350117 (2013).
- [31] G. Kofinas, R. Maartens and E. Papantonopoulos, JHEP **0310**, 066 (2003).
- [32] R.A. Brown, R. Maartens, E. Papantonopoulos and V. Zamarias, JCAP **0511**, 008 (2005).
- [33] S. Nojiri, S.D. Odintsov and M. Sasaki, Phys. Rev. D **71**, 123509 (2005).
- [34] S. Chaplygin, Sci. Mem. Moscow Univ. Math. Phys. **21**, 1 (1904).
- [35] A.Y. Kamenshchik, U. Moschella, and V. Pasquier, Phys. Lett. B **511**, 265 (2001).
- [36] M.C. Bento, O. Bertolami, and A.A. Sen, Phys. Rev. D **66**, 043507 (2002).
- [37] V. Gorini, A. Kamenshchik, U. Moschella, V. Pasquier; [arXiv:gr-qc/0403062].
- [38] Z.H. Zhu, Astron. Astrophys., **423**, 421 (2004).
- [39] M.C. Bento, O. Bertolami and A.A. Sen , Phys. Lett. B **575**, 172 (2003).
- [40] N. Bilic, G.B.Tupper and R.D. Viollier, Phys. Lett. B **535**, 17 (2001).
- [41] U. Debnath, A. Banerjee, and S. Chakraborty, Class. Quantum Grav. **21**, 5609 (2004).
- [42] R.A. Brown, Gen. Rel. Grav. **39**, 477 (2007).
- [43] R.G. Cai, H.S. Zhang and A. Wang, Commun. Theor. Phys. **44**, 948 (2005).
- [44] S. Nojiri, S. D. Odintsov, and S. Ogushi, Int. J. Mod. Phys. A **17**, 4809 (2002).
- [45] M. Malekjani, A. Khodam-mohammadi, Int. J. Mod. Phys. D **20**, 281 (2011).
- [46] M.C. Bento, O. Bertolami and A.A. Sen, Phys.Rev. D **70**, 083519 (2004) .
- [47] P.J.E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003).
- [48] D. Pavon *et al.*, Phys. Rev. D **86**, 083509 (2012).
- [49] A. Khodam-mohammadi, E. Karimkhani, Int. J. Mod. Phys. D **23**, 1450081 (2014).
- [50] R.A. Daly *et al.*, Astrophys. J. **677**, 1 (2008).
- [51] P.J. Steinhardt *et al.*, Phys.Rev. D **59**, 123504 (1999).
- [52] J. Yoo and Y. Watanabe, Int. J. Mod. Phys. D **21**, 1230002 (2012).